

On the congruence $x^x \equiv \lambda \pmod{p}$

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Abstract

In the present paper we obtain several new results related to the problem of upper bound estimates for the number of solutions of the congruence

$$x^x \equiv \lambda \pmod{p}; \quad x \in \mathbb{N}, \quad x \leq p-1,$$

where p is a large prime number, λ is an integer coprime to p . Our arguments are based on recent estimates of trigonometric sums over subgroups due to Shkredov and Shteinikov.

1 Introduction

For a prime p and an integer λ let $J(p; \lambda)$ be the number of solutions of the congruence

$$x^x \equiv \lambda \pmod{p}; \quad x \in \mathbb{N}, \quad x \leq p-1. \quad (1)$$

Note that the period of the function x^x modulo p is $p(p-1)$, which is larger than the range in congruence (1).

From the works of Crocker [4] and Somer [8] it is known that there are at least $\lfloor (p-1)/2 \rfloor$ and at most $3p/4 + p^{1/2+o(1)}$ incongruent values of $x^x \pmod{p}$ when $1 \leq x \leq p-1$. There are several conjectures in [5] related to this function.

New approaches to study $J(p; \lambda)$ were given by Balog, Broughan and Shparlinski, see [1] and [2]. In the special case $\lambda = 1$ it was shown in [1] that $J(p; 1) < p^{1/3+o(1)}$. This estimate was slightly improved in our work [3] to the bound $J(p; 1) \ll p^{1/3-c}$ for some absolute constant $c > 0$. Note that the method of [3] applies for a more general exponential congruences, however, the constant c there becomes too small. In the present paper we use a different approach and prove the following results.

Theorem 1. *The number $J(p; 1)$ of solutions of the congruence*

$$x^x \equiv 1 \pmod{p}; \quad x \in \mathbb{N}, \quad x \leq p-1, \quad (2)$$

satisfies $J(p; 1) \lesssim p^{27/82}$.

Here and below we use the notation $A \lesssim B$ to denote that $A < Bp^{o(1)}$; that is, for any $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that $A < cBp^\varepsilon$. As usual, $\text{ord } \lambda$ denotes the multiplicative order of λ , that is, the smallest positive integer t such that $\lambda^t \equiv 1 \pmod{p}$. We recall that $\text{ord } \lambda | p-1$.

Theorem 2. *Uniformly over $t | p-1$, we have, as $p \rightarrow \infty$,*

$$\sum_{\substack{1 \leq \lambda \leq p-1 \\ \text{ord } \lambda = t}} J(p; \lambda) \lesssim t + p^{1/3} t^{1/2}. \quad (3)$$

In the range $t < p^{1/3}$ our Theorem 2 improves some results of the aforementioned works [1] and [2]. Note that in the case $t = 1$ the estimate of Theorem 1 is stronger. In fact, following the argument that we use in the proof of Theorem 1 it is possible to improve Theorem 2 in specific small ranges of t .

Let now $I(p)$ denote the number of solutions of the congruence

$$x^x \equiv y^y \pmod{p}; \quad x \in \mathbb{N}, \quad y \in \mathbb{N}, \quad x \leq p-1, \quad y \leq p-1.$$

There is the following relationship between $I(p)$ and $J(p; \lambda)$:

$$I(p) = \sum_{\lambda=1}^{p-1} J(p; \lambda)^2.$$

We modify one of the arguments of [1] and obtain the following refinement on [1, Theorem 8].

Theorem 3. *We have, as $p \rightarrow \infty$,*

$$I(p) \lesssim p^{23/12}. \quad (4)$$

In order to prove our results, we first reduce the problem to estimates of exponential sums over subgroups. In the proof of Theorem 1 we use Shteinikov's result from [7], while in the proof of Theorem 2 we use Shkredov's result from [6] (see, Lemma 2 and Lemma 3 below).

In what follows, \mathbb{F}_p is the field of residue classes modulo p . The elements of \mathbb{F}_p we associate with their concrete representatives from $\{0, 1, \dots, p-1\}$. For an integer m coprime to p by m^* we denote the smallest positive integer such that $m^*m \equiv 1 \pmod{p}$. We also use the abbreviation

$$e_p(z) = e^{2\pi iz/p}.$$

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2 Lemmas

Lemma 1. *Let*

$$\lambda \not\equiv 0 \pmod{p}, \quad n \in \mathbb{N}, \quad 1 \leq M \leq p.$$

Then for any fixed constant $k \in \mathbb{N}$ the number J of solutions of the congruence

$$x^n \equiv \lambda \pmod{p}, \quad x \in \mathbb{N}, \quad x \leq M,$$

satisfies

$$J \lesssim \left(1 + \frac{M}{p^{1/k}}\right) n^{1/k}.$$

In particular, if $n = dt < p$ and $M = p/d$, then we have the bound

$$J \lesssim \left(d^{1/k} + \left(\frac{p}{d}\right)^{1-1/k}\right) t^{1/k}.$$

Proof. We have

$$J^k \lesssim \#\{(x_1, \dots, x_k) \in \mathbb{N}^k \cap [1, M]^k; \quad (x_1 \dots x_k)^n \equiv \lambda^k \pmod{p}\}.$$

Since for a given integer μ the congruence

$$X^n \equiv \mu \pmod{p}, \quad X \in \mathbb{N}, \quad X \leq p,$$

has at most n solutions, there exists a positive integer $\lambda_0 < p$ such that

$$J^k \lesssim nJ_1,$$

where J_1 is the number of solutions of the congruence

$$x_1 \dots x_k \equiv \lambda_0 \pmod{p}; \quad (x_1, \dots, x_k) \in \mathbb{N}^k \cap [1, M]^k.$$

It follows that

$$x_1 \dots x_k = \lambda_0 + py; \quad (x_1, \dots, x_k) \in \mathbb{N}^k \cap [1, M]^k, \quad y \in \mathbb{Z}.$$

Since the left hand side of this equation does not exceed M^k , we get that $|y| \leq M^k/p$. Hence, for some fixed y_0 we have

$$J_1 \lesssim \left(1 + \frac{M^k}{p}\right) J_2,$$

where J_2 is the number of solutions of the equation

$$x_1 \dots x_k = \lambda_0 + py_0; \quad (x_1, \dots, x_k) \in \mathbb{N}^k \cap [1, M]^k.$$

Hence, from the bound for the divisor function it follows that $J_2 \lesssim 1$. Thus,

$$J^k \lesssim \left(1 + \frac{M^k}{p}\right) n.$$

and the result follows. \square

Let H_d be the subgroup of $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ of order d . From the classical estimates for exponential sums over subgroups it is known that

$$\left| \sum_{h \in H_d} e_p(ah) \right| \leq p^{1/2}.$$

For a wide range of d this bound has been improved in a series of works. Here, we need the results due to Shteinikov [7] (see Lemma 2 below) and Shkredov [6] (see Lemma 3 below). They will be used in the proof of Theorem 1 and Theorem 2, respectively.

Lemma 2. *Let H_d be the subgroup of \mathbb{F}_p^* of order $d < p^{1/2}$. Then for any integer $a \not\equiv 0 \pmod{p}$ the following bound holds:*

$$\left| \sum_{h \in H_d} e_p(ah) \right| \lesssim p^{1/18} d^{101/126}.$$

Lemma 3. *Let H_d be the subgroup of \mathbb{F}_p^* of order $d < p^{2/3}$. Then for any integer $a \not\equiv 0 \pmod{p}$ the following bound holds:*

$$\left| \sum_{h \in H_d} e_p(ah) \right| \lesssim p^{1/6} d^{1/2}.$$

The following two results are due to Balog, Broughan and Shparlinski from [1] and [2].

Lemma 4. *Uniformly over $t|p-1$, we have, as $p \rightarrow \infty$,*

$$\sum_{\substack{1 \leq \lambda \leq p-1 \\ \text{ord } \lambda = t}} J(p; \lambda) \lesssim t + p^{1/2}.$$

Lemma 5. *Uniformly over $t|p-1$ and all integers λ with $\gcd(\lambda, p) = 1$ and $\text{ord } \lambda = t$, we have, as $p \rightarrow \infty$,*

$$J(p; \lambda) \lesssim pt^{-1/12}.$$

We also need the following lemma.

Lemma 6. *Let a, x be positive integers and let $d = \gcd(x, p-1)$. Then $a^d \equiv 1 \pmod{p}$.*

This lemma is well-known and the proof is simple. Indeed, if $\text{ind } a$ is indice of a with respect to some primitive root g modulo p , then,

$$x \cdot \text{ind } a \equiv 0 \pmod{(p-1)}.$$

Therefore, $d \cdot \text{ind } a \equiv 0 \pmod{(p-1)}$, whence $a^d \equiv 1 \pmod{p}$.

The following lemma is also well-known; see, for example, exercise and solutions to chapter 3 in Vinogradov's book [9] for even a more general statement.

Lemma 7. *For any integers U and $V > U$ the following bound holds:*

$$\sum_{a=1}^{p-1} \left| \sum_{z=U}^V e_p(az) \right| \lesssim p.$$

3 Proof of Theorem 1

We have

$$J(p; 1) = \sum_{d|p-1} J'_d,$$

where J'_d is the number of solutions of (2) with $\gcd(x, p-1) = d$. It then follows by Lemma 6 that

$$J(p; 1) \leq \sum_{d|p-1} J_d,$$

where J_d is the number of solutions of the congruence

$$z^d \equiv (d^d)^* \pmod{p}, \quad z \in \mathbb{N}, \quad z \leq (p-1)/d.$$

We have therefore,

$$J(p; 1) \leq R_1 + R_2 + R_3 + \sum_{\substack{d|p-1 \\ d < p^{3/7}}} J_d,$$

where

$$R_1 = \sum_{\substack{d|p-1 \\ d > p^{5/7}}} J_d; \quad R_2 = \sum_{\substack{d|p-1 \\ p^{4/7} < d < p^{5/7}}} J_d; \quad R_3 = \sum_{\substack{d|p-1 \\ p^{3/7} < d \leq p^{4/7}}} J_d.$$

The trivial estimate $J_d \leq p/d$ implies that

$$R_1 \lesssim \sum_{\substack{d|p-1 \\ d > p^{5/7}}} \frac{p}{d} \lesssim \sum_{d|p-1} p^{2/7} \lesssim p^{2/7}.$$

To estimate R_2 we use Lemma 1 with $k = 3$ and get

$$R_2 = \sum_{\substack{d|p-1 \\ p^{4/7} < d < p^{5/7}}} J_d \lesssim \sum_{\substack{d|p-1 \\ p^{4/7} < d < p^{5/7}}} (d^{1/3} + (p/d)^{2/3}) \lesssim \sum_{d|p-1} p^{2/7} \lesssim p^{2/7}.$$

To estimate R_3 we use Lemma 1 with $k = 2$ and get

$$R_3 = \sum_{\substack{d|p-1 \\ p^{3/7} < d < p^{4/7}}} J_d \lesssim \sum_{\substack{d|p-1 \\ p^{3/7} < d < p^{4/7}}} (d^{1/2} + (p/d)^{1/2}) \lesssim \sum_{d|p-1} p^{2/7} \lesssim p^{2/7}.$$

Thus,

$$J(p; 1) \lesssim p^{2/7} + \sum_{\substack{d|p-1 \\ d < p^{3/7}}} J_d.$$

Hence, there exists $d|p-1$ with $d < p^{3/7}$ such that

$$J(p; 1) \lesssim p^{2/7} + J_d. \quad (5)$$

Applying Lemma 1 with $k = 2$, we get

$$J_d \lesssim d^{1/2} + (p/d)^{1/2} \lesssim (p/d)^{1/2}. \quad (6)$$

Let now H_d be the subgroup of \mathbb{F}_p^* of order d . We recall that J_d is the number of solutions of the congruence

$$(dz)^d \equiv 1 \pmod{p}; \quad z \in \mathbb{N}, \quad z \leq (p-1)/d.$$

Therefore,

$$J_d = \#\{z \in \mathbb{N}; \quad z \leq (p-1)/d, \quad dz \pmod{p} \in H_d\}.$$

It then follows that

$$J_d = \frac{1}{p} \sum_{a=0}^{p-1} \sum_{1 \leq z \leq (p-1)/d} \sum_{h \in H_d} e_p(a(dz - h)).$$

Separating the term corresponding to $a = 0$ and using Lemma 2 for $a \neq 0$, we get

$$J_d \leq 1 + p^{1/18} d^{101/126} \left(\frac{1}{p} \sum_{a=1}^{p-1} \left| \sum_{1 \leq z \leq (p-1)/d} e_p(adz) \right| \right) \lesssim p^{1/18} d^{101/126}.$$

Using Lemma 7, we get the following bound for the double:

$$\sum_{a=1}^{p-1} \left| \sum_{1 \leq z \leq (p-1)/d} e_p(adz) \right| = \sum_{b=1}^{p-1} \left| \sum_{1 \leq z \leq (p-1)/d} e_p(bz) \right| \lesssim p.$$

Therefore

$$J_d \lesssim p^{1/18} d^{101/126}.$$

Comparing this estimate with (6) we obtain

$$J_d \lesssim p^{27/82}.$$

Incorporating this in (5), we get the desired result.

4 Proof of Theorem 2

In view of Lemma 4, it suffices to deal with the case $t < p^{1/3}$.

Since $\lambda^t \equiv 1 \pmod{p}$, it follows from (1) that

$$\sum_{\substack{1 \leq \lambda \leq p-1 \\ \text{ord } \lambda = t}} J(p; \lambda) \leq \#\{x \in \mathbb{N}; \quad x^{tx} \equiv 1 \pmod{p}, \quad x \leq p-1\}$$

Hence, denoting $d = \gcd(x, (p-1)/t)$ and using Lemma 6 we obtain that

$$\sum_{\substack{1 \leq \lambda \leq p-1 \\ \text{ord } \lambda = t}} J(p; \lambda) \leq \sum_{d|(p-1)/t} T_d,$$

where T_d is the number of solutions of the congruence

$$z^{dt} \equiv (d^{dt})^* \pmod{p}, \quad z \in \mathbb{N}, \quad z \leq (p-1)/d.$$

By the trivial estimate $T_d \leq p/d$ we have

$$\sum_{\substack{d|p-1 \\ d > p^{2/3}}} T_d \leq \sum_{d|p-1} p^{1/3} \lesssim p^{1/3}.$$

Furthermore, applying Lemma 1 with $k = 2$, we get

$$\sum_{\substack{d|p-1 \\ p^{1/3} < d < p^{2/3}}} T_d \leq \sum_{\substack{d|p-1 \\ p^{1/3} < d < p^{2/3}}} \left(d^{1/2} + (p/d)^{1/2} \right) t^{1/2} \lesssim p^{1/3} t^{1/2}.$$

Therefore,

$$\sum_{\substack{1 \leq \lambda \leq p-1 \\ \text{ord } \lambda = t}} J(p; \lambda) \leq p^{1/3} t^{1/2} + \sum_{\substack{d|(p-1)/t \\ d < p^{1/3}}} T_d. \quad (7)$$

Recall that $t < p^{1/3}$, thus $dt|p-1$ and $dt < p^{2/3}$.

Let H_{dt} be the subgroup of \mathbb{F}_p^* of order dt . Since T_d is the number of solutions of the congruence

$$(dz)^{dt} \equiv 1 \pmod{p}; \quad z \in \mathbb{N}, \quad z \leq (p-1)/d,$$

it follows that

$$T_d = \#\{z \in \mathbb{N}; \quad z \leq (p-1)/d, \quad dz \pmod{p} \in H_{dt}\}.$$

Therefore,

$$T_d = \frac{1}{p} \sum_{a=0}^{p-1} \sum_{1 \leq z \leq (p-1)/d} \sum_{h \in H_{dt}} e_p(a(dz - h)).$$

Separating the term corresponding to $a = 0$ and using Lemma 3 for $a \neq 0$ (with d replaced by dt), we get

$$T_d \leq t + p^{1/6} d^{1/2} t^{1/2} \left(\frac{1}{p} \sum_{a=1}^{p-1} \left| \sum_{1 \leq z \leq (p-1)/d} e_p(adz) \right| \right).$$

Applying Lemma 7 to the double sum, as in the proof of Theorem 1, we obtain for $d < p^{1/3}$ the bound

$$T_d \lesssim t + p^{1/6} d^{1/2} t^{1/2} \lesssim t + p^{1/3} t^{1/2}.$$

Thus,

$$\sum_{\substack{d|(p-1)/t \\ d < p^{1/3}}} T_d \leq \sum_{d|p-1} (t + p^{1/3} t^{1/2}) \lesssim t + p^{1/3} t^{1/2}.$$

Putting this into (7), we conclude the proof.

5 Proof of Theorem 3

We follow the arguments of [1] with some modifications. We have

$$I(p) = \sum_{\lambda=1}^{p-1} J(p; \lambda)^2 = \sum_{t|p-1} \sum_{\substack{1 \leq \lambda \leq p-1 \\ \text{ord } \lambda = t}} J(p; \lambda)^2.$$

It then follows that for some fixed order $t|p-1$ we have

$$I(p) \lesssim \sum_{\substack{1 \leq \lambda \leq p-1 \\ \text{ord } \lambda = t}} J(p; \lambda)^2.$$

We can split the range of $J(p; \lambda)$ into $O(\log p)$ dyadic intervals. Then, for some $1 \leq M \leq p$, we have

$$I(p) \lesssim |\mathcal{A}| M^2, \tag{8}$$

where $|\mathcal{A}|$ is the cardinality of the set

$$\mathcal{A} = \{1 \leq \lambda \leq p-1; \quad \text{ord } \lambda = t, \quad M \leq J(p; \lambda) < 2M\}.$$

From Lemma 5 we have

$$M \lesssim pt^{-1/12}. \quad (9)$$

On the other hand, by Lemma 4 we also have

$$|\mathcal{A}|M \lesssim \sum_{\lambda \in \mathcal{A}} J(p; \lambda) \lesssim \sum_{\substack{1 \leq \lambda \leq p-1 \\ \text{ord } \lambda = t}} J(p; \lambda) \lesssim t + p^{1/2}.$$

If $t < p^{1/2}$, then using (8) we get

$$I(p) \lesssim |\mathcal{A}|M^2 \lesssim (|\mathcal{A}|M)^2 \lesssim p,$$

and the result follows. If $t > p^{1/2}$, then we get $|\mathcal{A}|M \lesssim t$. Therefore, using (8) and (9) we get

$$I(p) \lesssim |\mathcal{A}|M^2 \lesssim t(pt^{-1/12}) = pt^{11/12} \lesssim p^{23/12}.$$

This proves Theorem 3.

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